Problem of Solidification with Newton's Cooling at the Surface

PETER HRYCAK

Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey

An approximate solution to the problem of freezing of an isotropic, semi-infinite slab is proposed for the situation where Newton's cooling at the surface of the slab applies. Comparison with the existing exact solution shows the approximate solution to be practical.

The heat-balance integral method is used with the assumption of linear temperature distribution in the frozen zone and of a parabolic temperature profile in the liquid zone. The final expression for the depth of solidification, however, seems to be only little affected by the particular temperature profiles chosen.

One of the more involved problems in heat conduction concerns moving boundaries and change of phase. This is the problem associated with the formation of ice, the freezing of moist soils, certain chemical reactions, and in general, with processes where moving heat sources or sinks occur together with temperature-dependent property values.

Well-known applied mathematicians of the past such as Lamé, Clapeyron, and Stefan (*1*) have concerned themselves with this problem, which recently has attracted a large number of investigators (*8*).

The principal difficulty here is caused by the nonlinearity of the problem. Brillouin (*1*), in a discussion of the particular solutions of a few special cases by Neumann and Stefan, points out the mathematical complications associated with obtaining analytical solutions for the less restricted boundary conditions. On the other hand, much use has been made lately of the technique of a simplified solution of heat-conduction equation by the so-called *integral* method associated with the name of Goodman (*3*). This method had been applied in a slightly different form, relatively long ago, to approximate solutions of equations of diffusion type (*10*) and, more recently, to the problem of frost penetration in moist soils (*9, 4*). Veinik (*11*) has written a complete book on a variation of the method with a chapter on applications to problems of metal solidification.

THE PROBLEM

This paper presents an application of this method to the case of freezing a semi-infinite, isotropic slab, originally at a constant temperature T_∞ throughout. At $t = t_o$ the slab is exposed to the cooling medium of a constant temperature, T_m. When Newton's cooling at the face ($x = 0$) applies, this results in the following boundary-value problem:

$$T(x,t) = T_\infty, \quad x \geqq 0, \quad t < t_o \qquad (1a)$$

$$-k_1 \frac{\partial T_1}{\partial x} = h(T_m - T_s) \quad t \geqq t_o, \quad x = 0 \qquad (1b)$$

$$T_1(\xi, t) = T_2(\xi, t) = T_f \qquad (1c)$$

$$-k_1 \frac{\partial T_1}{\partial x} = -k_2 \frac{\partial T_2}{\partial x} = -\frac{d\xi}{dt} L, \quad x = \xi, \qquad (1d)$$

The differential equations themselves are

$$\frac{\partial T_i}{\partial t} = a_i \frac{\partial^2 T_i}{\partial x_i}, i = 1, 2 \qquad (2)$$

Because of the nonlinear boundary condition, Equation (*1d*), the whole problem becomes nonlinear. As is ordinarily done with integral methods, the solution is assumed to have the form of a polynomial which is fitted to meet the boundary conditions, Equation (1).

Hence, the assumption will be made that after the beginning of the freezing process, the temperature distribution is linear in the frozen zone and parabolic in the liquid zone, such that for $T \leqq T_f$

$$T_1 = T_m + \frac{(T_f - T_m)(x + k')}{(\xi + k')}, \quad 0 < x < \xi \qquad (3a)$$

whereas

$$T_1 = T_f, \quad x \geqq \xi$$

Also, for $T \geqq T_f$

$$T_2 = T_f + 2(T_\infty - T_f)\left[\frac{x-\xi}{x_t - \xi} - \frac{1}{2}\left(\frac{x-\xi}{x_t-\xi}\right)^2\right], \quad \xi < x < x_t \qquad (3b)$$

and

$$T_2 = T_\infty, \quad x \geqq x_t$$

where x_t signifies the depth to which the effects of surface temperature have penetrated.

This particular temperature profile is assumed here mainly for reasons of convenience, but it can be justified by the fact that in the frozen zone a nearly linear tem-

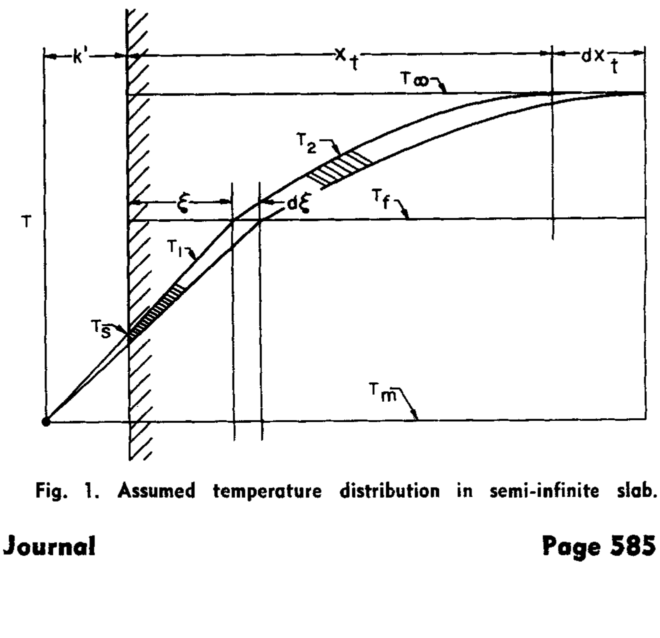

Fig. 1. Assumed temperature distribution in semi-infinite slab.

perature distribution will result if the effect of the sensible heat of that zone is relatively small in comparison with the combined effects of the latent heat and the sensible heat of the liquid zone. In the liquid zone it would be possible to introduce higher order polynomials by equating the corresponding derivatives to zero. Consequently cubic, quartic, and a quarter sine wave profiles were tried, but the results obtained did not show any significant improvement. As may be seen from Figure 1, Equations (3*a*) and (3*b*) satisfy the boundary conditions, except that the $T = T_\infty$ type already applies for $x = x_t$, instead of being the limiting value as $x \to \infty$. This distortion of the temperature profile is characteristic of the method used. The value of x_t is calculated from the consideration that all the heat that results from the cooling process must be exactly equal to the energy going through the surface of the slab. This leads to the equation

$$\frac{d}{dt}\left[C_1\int_0^\infty T_1\,dx + C_2\int_\xi^\infty (T_2 - T_f)\,dx + L\int_\xi^\infty dx\right] = -k_1\frac{\partial T_1}{\partial x}\bigg|_{x=0} \quad (4)$$

Equation (4) may also be obtained directly from integration of the heat equation itself with respect to x (3), with an adjustment for latent heat effects. If Equations (3*a*) and (3*b*) are substituted in Equation (4), with $\theta_1 = T_f - T_m$ and $\theta_2 = T_\infty - T_f$, there results

$$\left[\theta_1 C_1\left(\frac{1}{2}\xi^2 + k'\xi\right)(\xi + k')^{-2} + \theta_2 C_2\left(\frac{1}{3}\frac{dx_t}{d\xi} + \frac{2}{3}\right) + L\right]d\xi = k_1\theta_1\,(\xi + k')^{-1}\,dt \quad (5)$$

Equation (5) can be simplified by introducing the concept of the dimensionless (reciprocal) latent heat μ and of the dimensionless sensible heat α and may be rearranged to read

$$\left\{\frac{1}{2}\mu\left[1 - \left(\frac{\xi}{k'} + 1\right)^{-2}\right] + \alpha\mu\left(\frac{1}{3}\frac{dx_t}{d\xi} + \frac{2}{3}\right) + 1\right\}L\,d\xi = k_1\theta_1\,(\xi + k')^{-1}\,dt \quad (6)$$

If it is assumed that $dx_t/d\xi \approx M$, a constant, then Equation (6) may be integrated at once, giving

$$\left[\frac{1}{2}\mu + \alpha\mu\left(\frac{1}{3}M + \frac{2}{3}\right) + 1\right]\left[\left(\frac{1}{2}\xi^2 + k'\xi\right) - \left(\frac{1}{2}\xi_o^2 + k'\xi_o\right)\right] + \frac{1}{2}\mu(k')^2 \ln\left[\frac{(\xi_o + k')}{(\xi + k')}\right] = k_1\theta_1\,(t - t_o)\,L^{-1} \quad (7)$$

considering that for $t = t_o$, $\xi = \xi_o$.

Equation (7) may be written

$$\nu^{-2}\left[\left(\frac{1}{2}\xi^2 + k'\xi\right) - \left(\frac{1}{2}\xi_o^2 + k'\xi_o\right)\right] + \frac{1}{2}\mu(k')^2 \ln\left[\frac{(\xi_o + k')}{(\xi + k')}\right] = k_1\theta_1\,(t - t_o)\,L^{-1} \quad (7a)$$

with ν^{-2} replacing the terms in the brackets in the first member on the left-hand side in Equation (7); ν is a new parameter to be determined later. Equation (7*a*) may be solved for ξ according to the quadratic formula

$$\xi = -(k') + \left\{(\xi_o + k')^2 + \nu^2\left[\frac{2k_1\theta_1}{L}(t - t_o) - \mu(k')^2\ln\left[\frac{(\xi_o + k')}{(\xi + k')}\right]\right]\right\}^{1/2} \quad (8)$$

For the case considered in this paper the slab is originally at the temperature T_∞; consequently, $\xi_o = 0$ applies. For simplicity, one can take also $t_o = 0$. Equation (8) is the basic expression for the depth of solidification, ξ; it can be solved for ξ by iteration for sufficiently small values of k', if ν is known. The difficulty lies in determining the value of ν.

DEFINITION AND CALCULATION OF ν

If the term $(\xi/k' + 1)^{-2}$ that occurs in Equation (6) is set equal to ϵ, one obtains $0 < \epsilon \leq 1$, with $\epsilon = 1$ for $\xi = 0$ and $\epsilon \to 0$ as $\xi \to \infty$ for a given k'. Then Equation (6) may be written

$$\left[\frac{1}{2}\mu\,(1 - \epsilon) + \alpha\mu\left(\frac{1}{3}M + \frac{2}{3}\right) + 1\right]L\,d\xi = k_1\theta_1\,(\xi + k')^{-1}\,dt \quad (9)$$

from which one can define

$$\frac{1}{2}\mu\,(1 - \epsilon) + \alpha\mu\left(\frac{1}{3}M + \frac{2}{3}\right) + 1 = \nu_\epsilon^{-2} \quad (10)$$

ν_ϵ being analogous to ν introduced before, except for the factor $(1 - \epsilon)$ in the first member on the left-hand side; consequently, ν may be considered to be the limit of ν_ϵ as $\epsilon \to 0$. The term ν_ϵ stands for the relative contribution of the sensible heat of the slab to the slowing down of the freezing process. From inspection of Equation (10), it is seen that if the slab is already at the freezing temperature at the time t_o (that is, if $\theta_2 = 0$, equivalent to $\alpha = 0$), and if the term due to sensible heat of the frozen zone is neglected in comparison with unity ($\mu = 0$), $\nu_\epsilon = \nu = 1$ results.

Therefore,

$$\frac{d\xi}{dt} = k_1\theta_1\frac{\nu_\epsilon^2}{L}(\xi + k')^{-1} \quad (11)$$

With Equations (3*a*) and (3*b*), one has, according to Equation (1*d*), at the interface between the two zones at $x = \xi$

$$\frac{k_1\theta_1}{\xi + k'} = 2\frac{k_2\theta_2}{x_t - \xi} + L\frac{d\xi}{dt} \quad (12)$$

When Equation (11) is substituted into Equation (12), rearranged and differentiated, and $d\nu_\epsilon/d\xi \approx 0$ is set, one obtains

$$dx_t - d\xi = 2\frac{\alpha}{\delta^2}(1 - \nu_\epsilon^2)^{-1}\,d\xi \quad (13)$$

Therefore, when $\nu_\epsilon \approx$ const, also $dx_t/d\xi \approx \text{const} = M$, as has been assumed previously. The term δ^2 is the dimensionless thermal diffusivity. When Equations (10) and (13) are combined, one obtains

$$\frac{1}{\nu_\epsilon^2} = 1 + \frac{1}{2}(1 - \epsilon)\,\mu + \alpha\mu + \frac{2}{3}\frac{\alpha^2\mu}{\delta^2}\frac{1}{1 - \nu_\epsilon^2} \quad (14)$$

which can be rearranged to read

$$\nu_\epsilon^4\left[1 + \frac{1}{2}(1 - \epsilon)\,\mu + \alpha\mu\right] - \nu_\epsilon^2\left[2 + \frac{1}{2}(1 - \epsilon)\,\mu + \alpha\mu + \frac{2}{3}\frac{\alpha^2\mu}{\delta^2}\right] + 1 = 0 \quad (15)$$

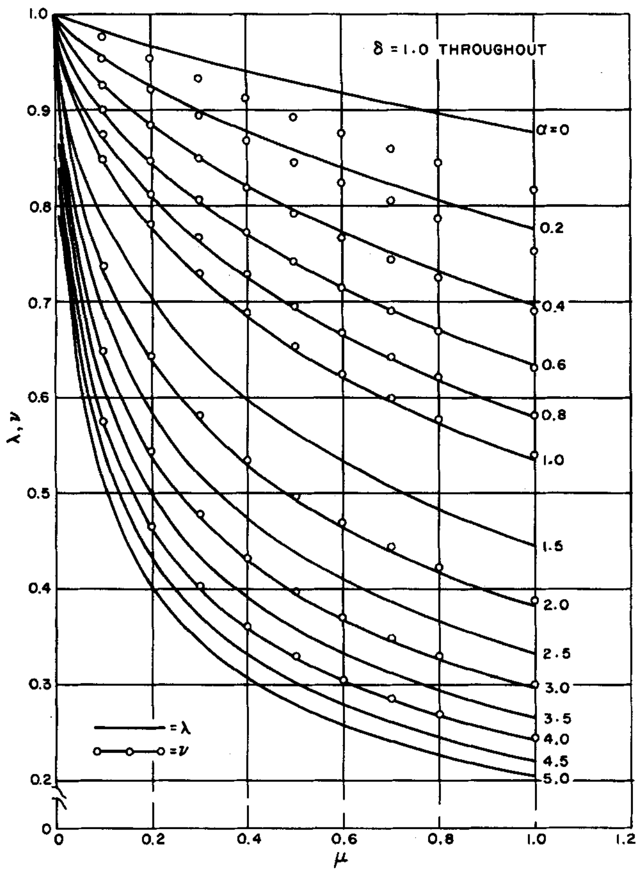

Fig. 2. Comparison of λ and ν, δ = 1.0, as a function of α and μ.

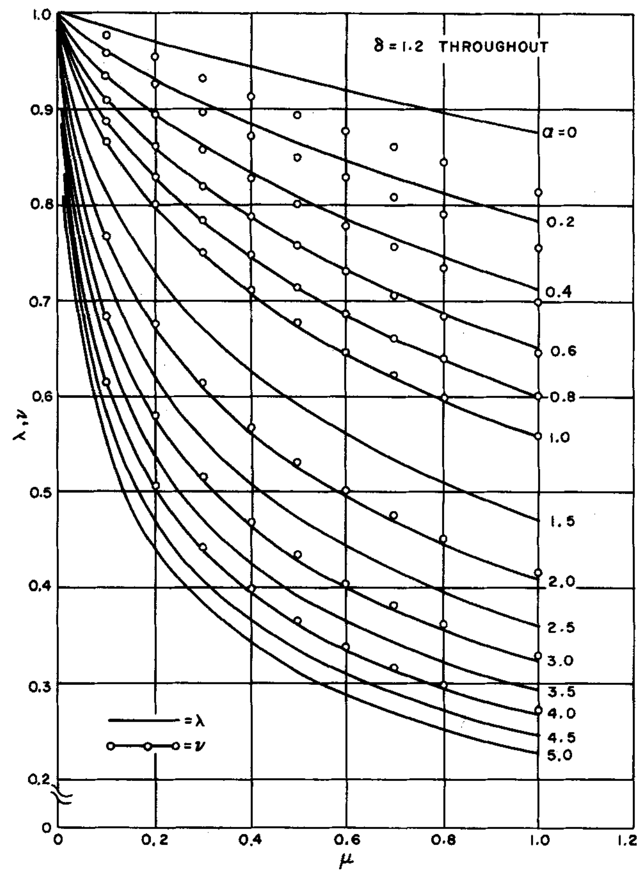

Fig. 3. Comparison of λ and ν, δ = 1.2, as a function of α and μ.

Equation (15) can be readily solved for any value of the parameters occurring in it. The solution is uniquely defined from the consideration that $0 < \nu_\epsilon \leqq 1$. Although the parameters α, μ, and δ are related to property values only, ϵ is also a function of time. In particular, since the ratio $h\xi/k_1$ is known as the Biot number, $Bi(\xi)$

$$\epsilon = (1 + Bi(\xi))^{-2} \tag{16}$$

In general, the Biot number is an indicator of surface heat transfer intensity. For large values of the Biot number, $\epsilon \approx 0$ applies. For initially small values of $Bi(\xi)$, ν_ϵ may be calculated in steps, taking for each step a separate value of ϵ based on the corresponding range of ξ values from Equation (8). As ξ increases, on the other hand, $\epsilon \to 0$, $\nu_\epsilon \to f(\alpha, \delta, \mu) = \nu$.

COMPARISON WITH THE EXACT SOLUTION

It is interesting to note that for $h \to \infty$, the temperature at the surface of the slab becomes equal to that of the cooling medium, or $T_s = T_m$. An exact solution that is due to Neumann (3, p. 112), is available for this situation. It gives for the depth of frost penetration the expression

$$\xi = \lambda \sqrt{\frac{2k_1\theta_1 t}{L}} \tag{17}$$

where λ is the root of the transcendental equation

$$\frac{1}{\sqrt{\pi}} \frac{\exp\left(-\frac{1}{2}\lambda^2\mu\right)}{\operatorname{erf} \lambda\sqrt{\frac{\mu}{2}}}$$

$$-\frac{1}{\sqrt{\pi}}\frac{\alpha}{\delta}\frac{\exp\left(-\frac{1}{2}\lambda^2\delta^2\mu\right)}{\operatorname{erfc}\left(\lambda\delta\sqrt{\frac{\mu}{2}}\right)} = \lambda\frac{1}{\sqrt{2\mu}} \tag{18}$$

and so $\lambda = g(\alpha, \delta, \mu)$.

From Equation (8) it is seen that for $h \to \infty$ and $t_o = \xi_o = 0$

$$\xi = \nu \sqrt{\frac{2k_1\theta_1 t}{L}} \tag{19}$$

which is identical to Equation (17), except that the approximate coefficient, ν, is used instead of the exact one, λ. Furthermore, from Equation (15) and the subsequent discussion, the relation $\nu = f(\alpha, \delta, \mu)$ applies. Thus, both ν and λ are shown to be the functions of the same three parameters, α, μ, and δ. The defining equation for λ is transcendental and can be solved only numerically, whereas the value of ν may be obtained from the relatively easy-to-solve Equation (15). Both λ and ν are plotted in Figures 2 and 3 as functions of α, μ, and δ, and good agreement is found, except for the region of $\alpha = 0$.

This is so, because for $\alpha = 0$ the effects of thermal capacity of the frozen zone become more significant. Then the expression for the sensible heat of the frozen zone is no longer adequately represented by the total heat balance based on a straight-line temperature profile as given here by Equation (3*a*). However, if the coefficient of the term $(1 - \epsilon)\ \mu$ is set in Equation (15) equal to 1/3 (this would correspond roughly to the heat balance based on a parabolic profile) instead of 1/2 (which is, strictly speaking, applicable to only a straight line profile), the sensible heat of the frozen zone is better accounted for and a better accuracy is realized. In this case, $\nu = \left(1 + \frac{1}{3}\mu\right)^{-1/2}$ applies, identical to the first-order approximation for λ in the exact handling of the problem (*12*). Then, for example, for $\mu = 0.3$, the value of ν changes from 0.9325 to 0.9535 whereas the exact value is $\lambda = 0.9550$. The case of $\alpha = 0$ is of interest primarily in the problems of metal solidification; it has been quite thoroughly treated by Veinik (*11*, p. 137 ff.). On the other hand, as has been mentioned already, the assumption of a different temperature profile in the liquid zone influences the ν values only very slightly.

COMPARISON WITH STEFAN'S SIMPLIFIED SOLUTION

From an inspection of Equation (15) it is seen that its root is actually $\nu = 1$ if $\alpha = \mu \rightarrow 0$ and that $\nu \rightarrow 0$ as $\alpha \rightarrow \infty$ or $\mu \rightarrow \infty$. In the first case, with $\nu = 1$, the situation is equivalent to one where the effects of sensible cooling are disregarded altogether. The solution of this kind, for $h \rightarrow \infty$

$$\xi = \sqrt{\frac{k_1 \theta_1 t}{L}} \tag{20}$$

is associated with the name of Stefan (*5*). In the literature it is often suggested for an approximate calculation of the depth of frost penetration (*2*, p. 113). The second case, with $\nu = 0$, would have the physical significance of a dry medium with the latent heat $L \rightarrow 0$ so that freezing may not take place at all. A modified version of Stefan's solution will be obtained if in Equation (8) the effects of sensible heat are disregarded so that $\alpha = \mu = 0$; then, $\nu = 1$ applies and one obtains

$$\xi = -k' + \left[(\xi_o + k')^2 + \frac{2}{L} k_1 \theta_1 (t - t_o) \right]^{1/2} \tag{21}$$

an approximate expression which includes the effect of the surface film coefficient, h.

Equation (21) may be used for getting quick estimates of maximum depth of frost penetration under conditions where there is a considerable difference between the temperature of the cooling medium, T_m, and that of the surface of the slab, T_s.

JUSTIFICATION OF THE METHOD

The solution for the depth of freezing, Equation (8), depends on the method of determining the auxiliary parameter ν from Equation (15). Even if this method of getting ν is considered to be a mere duplication of the existing exact solution, it is convenient for calculating approximately the parameter λ in Equation (17) for values of α, μ, and δ that have not been tabulated.

Also, the treatment of the problem by approximate integral methods leads to additional approximations which are more accurate than the ones that are based merely on some form of Equation (20), with effects of sensible cooling disregarded. Examples of this are found in reference *4*.

Certain assumptions necessary to derive the formula for frost penetration, Equation (8), can be justified only if the value of ξ is found, consequently, to be little affected. This can be done by consideration of the upper and the lower bounds on ξ as a function of the Biot number and by comparison with the exact solution for the limiting case of $Bi \rightarrow \infty$ associated with $h \rightarrow \infty$.*

In the following paragraphs examples will be given of how the finite values of surface film coefficients compare with the case of $h \rightarrow \infty$.

* See Appendix A for details.

One can take here for the finite h values, $h = 6.00$ and $h = 1.65$, both in British thermal units per hour per square foot, which are considered representative values for building materials (*6*). The higher value corresponds to outdoor conditions and the lower one to typical inside still air conditions.

An example is the freezing of a soil rich in silt and clay, with a dry specific weight ρ of 90 lb./cu. ft. and a moisture content u of 0.2 lb. water/lb. soil. For thermal properties of such a soil, one finds in the tables prepared by Kersten (*7*), $k_1 = 0.83$, $k_2 = 0.641$, both in British thermal units per hour per foot per degree Fahrenheit. One also finds $C_1 = \rho(0.17 + 0.5u)$ and $C_2 = \rho(0.17 + u)$ for a typical soil; also, one can take $L = \rho\ 143.5u$. Then the numerical values of these quantities become $C_1 = 24.3$, $C_2 = 33.3$, both in British thermal units per degree Fahrenheit per cubic foot, and $L = 2{,}580$ B.t.u./cu. ft.

In a case where soil, originally at $T_\infty = 50$°F., is frozen by exposure to the air at $T_m = -20$°F., the depth of freezing will be calculated from Equation (8). The fact that the fictitious added slab thickness k' (Figure 1) has no sensible heat is included in Equation (8) in the form of the log term; if this term is disregarded, it is equivalent to treating the slab as having its surface at $x = -k'$ reduced suddenly at $t = t_o$ to the temperature of the medium T_m, with $\xi_o = -k'$. This modified version of Equation (8) can be regarded to be the lower bound for the calculated depth of freezing. The upper bound will be, of course, the exact Neumann's solution, Equation (17).

In this problem the parameters necessary for calculating the value of ν become $\alpha = 0.475$, $\mu = 0.490$, and $\delta = 1.33$ and so from Equation (15) for $\epsilon = 0$ (or from an extrapolation from Figures 2 and 3) one gets $\nu = 0.793$. The choice of $\epsilon = 0$ is justified by the presence of bounds on Equation (15) that apply for $\epsilon \equiv 0$. The times of freezing will be taken as $t = 1$ day and $t = 100$ days, respectively. The results are tabulated in Table 1. From Table 1 it is seen from comparison with Neumann's solution that, as the time of freezing increases, the effect of a finite h decreases, but for relatively short freezing times the effect can be considerable.

Also, from the comparison of the solution by Equation (8) with the modified form of this equation as the lower bound, one sees that the difference is not very large. The figures given by Equation (21) indicate that, for a wide range of h in the freezing process, the effects of sensible cooling predominate over those of surface heat transfer except for relatively short freezing times.

The above example was chosen because the magnitude of the removable sensible heat of the frozen zone was relatively large in comparison with that of the latent heat. This is also the case where the effects of approximations

TABLE 1. COMPARISON OF FORMULAS FOR CALCULATION OF DEPTH OF FROST PENETRATION

Value of surface film coefficient	Time, days	Lower bound, mod. Eq. (8)*, ft.	Eq. (8), ft.	Upper bound, Eq. (19), ft.	Sensible heat effects disregarded, Eq. (21), ft.
$h = \infty$	1	0.711	0.711	0.711	0.897
	100	7.11	7.11	7.11	8.97
$h = 6.00$	1	0.586	0.594	0.711	0.769
	100	6.97	6.98	7.11	8.83
$h = 1.65$	1	0.368	0.394	0.711	0.524
	100	6.63	6.64	7.11	8.48

* With the log term disregarded.

necessary in the derivation of Equation (8) can be expected to be quite significant. Despite this fact, the results shown in Table 1 seem to be satisfactory for practical purposes.

CONCLUSIONS

The heat-balance integral method has been applied here to a problem related to heat conduction with a change of phase. The subject chosen was well within the area of applicability of integral methods, which is primarily the case of parabolic differential equations, with the simple initial and monotonic boundary conditions.

Since, in the present application of the method, it was necessary to assume a definite temperature profile beforehand to get a solution, it appears most logical to use such solutions mainly for calculation of quantities that depend on temperature only weakly and indirectly. Because the results discussed in the paper are concerned to a large extent with the latent heat effects, they, in fact, depend only weakly upon the exact temperature distribution in the slab and are little affected by the particular temperature profile chosen. Consequently, calculation of the depth of freezing in moist soils or of the depth of solidification in metals seems to be particularly well suited to integral methods, according to this point of view. And, as has been already pointed out by Veinik (*11*) such applications of the integral method result in very good accuracy for cases that are readily verifiable.

This would be analogous to the situation in fluid dynamics, where the integral methods have been used mainly to get approximate expressions for the thickness of the boundary layer (*2*). Similarly, in other branches of mechanics, methods essentially similar to integral methods, discussed here, have been used with success to calculate eigenvalues of difficult-to-solve differential equations.

NOTATION

a = thermal diffusivity, sq. ft./hr.
Bi = Biot number, $h\xi/k_1$
C = volumetric specific heat, B.t.u./(cu. ft. °F.)
h = surface film coefficient, B.t.u./(hr. sq. ft. °F.)
k = thermal conductivity, B.t.u./(hr. ft. °F.)
k' = k_1/h, ft.
L = volumetric latent heat of fusion, B.t.u./cu. ft.
t = time, hr.
t_o = time when cooling starts
T = temperature, °F.
u = moisture content, lb. water/lb. dry soil
x = depth, ft.
x_t = depth to which effects of surface cooling penetrated at the time t, ft.

Greek Letters

α = dimensionless sensible heat, $C_2\theta_2/C_1\theta_1$
δ = square root of the dimensionless thermal diffusivity, $(a_1/a_2)^{1/2}$
ϵ = $(\xi/k' + 1)^{-2}$
θ_1 = $T_f - T_m$, reduced temperature, frozen zone, °F.
θ_2 = $T_\infty - T_f$, reduced temperature, liquid zone, °F.
λ = root of Equation (18)
μ = ratio of the sensible heat of the frozen zone to the latent heat
ν = root of Equation (15)
ξ = depth of solidification, ft.
ξ_o = depth of solidification, corresponding to t_o, ft.
ρ = specific weight of dry soil, lb./cu. ft.

Subscripts

1 = frozen zone
2 = liquid zone
∞ = conditions far away from surface
f = condition where phase changes
m = condition of the cooling medium
s = conditions of the surface

APPENDIX A

It is claimed here that ξ from Equation (8) has a lower bound, ξ_l, obtained from the equation

$$\xi_l = -k' + \left[(\xi_o + k')^2 + \frac{\nu^2 2k_1\theta_1(t-t_o)}{L} \right]^{1/2}$$

and an upper bound, ξ_u, from Equation (17), or

$$\xi_l < \xi < \xi_u$$

for conditions where $\xi_o = t_o = 0$. This restriction is necessary because the exact solution is applicable only for $\xi_o = 0$. Then, when $-k'$ is subtracted from each member of the above inequalities, and one divides by ξ_u, and squares, and lets $k'/\xi_u = \eta$, one finds

$$\eta^2 + \nu^2\lambda^{-2} < \eta^2 [1 + \nu^2\mu \ln (\eta^{-1} + 1)] + \nu^2 \lambda^{-2} < \eta^2 + 2\eta + 1$$

It has been shown in Figures 2 and 3 that the relation $\nu^2\lambda^{-2} \approx 1$ for $\alpha \neq 0$ is true. Then, since $\eta > 0$ and $\nu < 1$, the above inequalities will be preserved if the condition $\eta\mu \ln (\eta^{-1} + 1) < 2$ is satisfied, or $\eta^{-1} + 1 < e^2\mu^{-1}\eta^{-1}$.

This condition will be true for all positive η and for all $0 < \mu < 2$. When $t \to \infty$, $\xi \to \infty$ and $\eta \to 0$. Then since

$$\lim_{\eta\to 0} \eta^2 \ln (\eta^{-1} + 1) = 0$$

the condition $\xi_l = \xi = \xi_u$ will apply.

Consequently, from the above reasoning the conclusion can be drawn that for $\nu \approx \lambda$, ξ from Equation (8) approaches the exact solution of the problem asymptotically from below as $t \to \infty$. This has the practical consequence that, for sufficiently long freezing times, the relative importance of heat transfer intensity at the surface of the slab becomes of little significance and the exact analytical solution of the problem may be used safely for finite values of the film coefficient, h.

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